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NEW GENERAL AND COMPLEMENTARY ENERGY THEOREMS, FINITE STRAIN, RATE-SENSITIVE INELASTICITY, AND FINITE ELEMENTS: SOME COMPUTATIONAL STUDIES

by

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New General and Complementary Energy Theorems, Finite Strain, Rate-Sensitive Inelasticity, and Finite Elements: Some Computational Studies.

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Abstract:

General variational theorems for the rate problems of rate-dependent finite strain inelasticity, in terms of the appropriate rates of the first and second Piola-Kirchhoff stress tensors, the symmetrized Biot-Lure' stress tensor, and their conjugate measures of strain-rate, are discussed. Certain new rate-complementary-energy principles, involving the rate of spin and the rate of the symmetrized Biot-Lure' stress tensor as variables, are stated for finite strain analysis of rate-sensitive materials, such as those exhibiting elasto-visco-plastic and creep behavior. Uniqueness and stability criteria for those inelastic solids, using the finite element counterparts of the new complementary energy rate principles, are discussed. Computational studies, using the complementary energy methods, discussed herein include: (*i*) bifurcation necking and post-buckling analyses of initially perfect elasto-plastic bars, and (*ii*) post-buckling and large-deformation analyses of thin elastic plates under inplane compression and transverse bending loads.

Introduction:

The topic of rate (incremental), multi-field, variational principles, in general, and the rate complementary energy principles, in particular, and the corresponding finite element methods, for finite strain analysis of compressible non-linear-elastic solids were discussed in detail by Atluri and Murakawa [1]. Also discussed in [1] were the contributions of Koiter, Zubov, and Fraeijs de Veubeke, dealing with the sub-

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ject of complementary energy principles, governing the *total* deformations of semi-linear and/or nonlinear compressible isotropic elastic materials. It was shown in [1], that the concept of treating the angular momentum balance condition as an a posteriori constraint through a complementary energy principle involving the symmetrized Biot-stress (or what is also referred to as the symmetrized Lure'-stress or the Jaumann-stress) as well as the orthogonal tensor of rigid rotation, as variables, as first introduced by F. de Veubeke, has certain fundamentally novel features that makes it attractive for practical application.

The ideas of discretizing the angular momentum balance conditions through a complementary energy principle has been extended by the authors in (*i*) the incremental (rate) analysis of finite strains in compressible as well as incompressible nonlinear elastic materials [2-4], (*ii*) the rate problems of classical (rate-independent) finite strain, elasto-plasticity [5-9], and (*iii*) nonlinear stability and post-bifurcation analysis of semilinear isotropic elastic beams [10].

In the present paper the authors' earlier work, [2-10], is extended to the cases of finite strain analyses of materials with rate-sensitive behavior such as elasto-viscoplasticity and creep, and post-buckling and large-deformation behavior of structural members such as plates and shells, undergoing large rotations and large stretches.

The summary of the topics presented in the following is: (*i*) a discussion of general (multi-field) variational principles, with emphasis on complementary energy, in terms of alternate stress-rates and conjugate measures of strain-rate, for rate-sensitive inelastic materials, (*ii*) rate complementary energy potentials, for the chosen stress-rates, for rate-dependent as well as rate-independent materials, (*iii*) criteria for uniqueness and stability of solutions, (*iv*) numerical study of necking of an initially perfect elasto-plastic bar, and (*v*) numerical study of post-buckling of an axially compressed plate of semilinear isotropic elastic material, undergoing large rotations, as well that of a thin plate undergoing large displacements due to transverse loading.

PRELIMINARIES:

We use a fixed rectangular cartesian coordinate system, and employ the notation: $(\underset{\sim}{\ })$ denotes a second-order tensor; $(\underset{\approx}{\ })$ denotes a fourth-order tensor; $(-)$ implies a vector; $\underline{a} = \underset{\sim}{A}.\underline{b}$ implies $a_i = A_{ij}b_j$; $\underset{\sim}{A}.\underset{\sim}{b}$ implies a product such that $(\underset{\sim}{A}.\underset{\sim}{B})_{ij} = A_{ij}B_{jk}$; $\underset{\sim}{A}:\underset{\sim}{B} = \text{trace}\,(\underset{\sim}{A}^T.\underset{\sim}{B}) = A_{ij}B_{ij}$; and $\underline{u}.\underline{t} = u_i t_i$. A particle in the undeformed body has a position vector $\underline{x} = x_\alpha \underline{e}_\alpha (\alpha=1..3)$ where $\underline{e}_\alpha$ are unit cartesian bases. The gradient operator $\underline{\nabla}^o$ in the undeformed configuration C_o is $\underline{\nabla}^o = (\underline{e}_\alpha \partial/\partial x_\alpha)$. The position vector of the particle in the deformed configuration, say C_N, is $\underline{y} = y_i\underline{e}_i$. The gradient operator in C_N is $\underline{\nabla}^N = (\underline{e}_i \partial/\partial y_i)$. The deformation gradient tensor is $\underset{\sim}{F} \equiv (\underline{\nabla}^o\underline{y})^T$, such that $F_{i\alpha} = y_{i,\alpha} \equiv (\partial y_i/\partial x_\alpha)$. The nonsingular $\underset{\sim}{F}$ has the polar-decomposition, $\underset{\sim}{F} = \underset{\sim}{\alpha}.(\underset{\sim}{I}+\underset{\sim}{h})$ where the rotation $\underset{\sim}{\alpha}$ is orthogonal and the stretch $\underset{\sim}{h}$ is symmetric and +ve definite. The Green-Lagrange strain is $\underset{\sim}{g} = 1/2(\underset{\sim}{F}^T.\underset{\sim}{F}-\underset{\sim}{I}) = 1/2(\underset{\sim}{e}+\underset{\sim}{e}^T+\underset{\sim}{e}^T.\underset{\sim}{e})$ where $\underset{\sim}{e} = (\underline{\nabla}^o\underline{u})^T$, $\underline{u} = \underline{y}-\underline{x}$.

For the present purposes, we introduce the stress measures (*i*) the "true" Cauchy stress $\underset{\sim}{\tau}$; (*ii*) a weighted tensor, the Kirchhoff stress tensor $\underset{\sim}{\sigma} = J\underset{\sim}{\tau}$ where J is determinant of matrix $[y_{i,\alpha}]$; (*iii*) the first Piola-Kirchhoff stress tensor $\underset{\sim}{t}$; (*iv*) the second Piola-Kirchhoff stress tensor $\underset{\sim}{s}$, and (*v*) the symmetrized Biot stress tensor (or what is also often referred to as the symmetrized Lure', or the Jaumann stress tensor) $\underset{\sim}{r}$. As discussed in [1,5], and elsewhere, the above stress measures are related as:

$$\underset{\sim}{\tau} = \frac{1}{J}\underset{\sim}{F}.\underset{\sim}{t} = \frac{1}{J}\underset{\sim}{F}.\underset{\sim}{s}.\underset{\sim}{F}^T = \frac{1}{J}\underset{\sim}{\sigma} \tag{1}$$

$$\underset{\sim}{t} = \underset{\sim}{s}.\underset{\sim}{F}^T = J(\underset{\sim}{F}^{-1}.\underset{\sim}{\tau});\quad \underset{\sim}{s} = J(\underset{\sim}{F}^{-1}.\underset{\sim}{\tau}.\underset{\sim}{F}^{-T}) \tag{2}$$

$$\underset{\sim}{r} = \frac{1}{2}(\underset{\sim}{t}.\underset{\sim}{\alpha}+\underset{\sim}{\alpha}^T.\underset{\sim}{t}^T) = \frac{1}{2}[\underset{\sim}{s}.(\underset{\sim}{I}+\underset{\sim}{h}) + (\underset{\sim}{I}+\underset{\sim}{h}).\underset{\sim}{s}] \tag{3}$$

The tensors $\underset{\sim}{\tau}$, $\underset{\sim}{\sigma}$, $\underset{\sim}{s}$, and $\underset{\sim}{r}$ are symmetric, while $\underset{\sim}{t}$ is unsymmetric. In the above, $\underset{\sim}{F}^{-T} = (\underset{\sim}{F}^{-1})^T$ and the superscript T denotes a transpose.

RATE FORMULATIONS:

Now, we consider the (incremental) rate analysis of finite strain problems of an inelastic solid with a rate-sensitive constitutive law. In doing so, one can choose an arbitrary reference frame. In practice, however, two choices, one the so-called total-Lagrangean (TL) and the other, the so-called updated-Lagrangean (UL) reference frames are appealing. Eventhough the choice of a reference frame does not, per se, affect the theoretical or computational approaches, we discuss the details of a UL formulation, since the rate consitutive relations of an inelastic solid depend, naturally, on the current state of true stress.

In the UL formulation, the solution variables in the generic state C_{N+1} are referred to the configuration of the body in the immediately preceding state, C_N, which is known. In the UL formulation, one is essentially concerned with an initial stress problem: the initial "true" stress in C_N is the Cauchy stress $\boldsymbol{\tau}^N$, while the initial displacements in C_N as referred to C_N are, obviously, zero. Let y_i^N be the current spatial coordinates of a particle in C_N. Let $\boldsymbol{\nabla}^N$ be the gradient operator in C_N (ie., $\boldsymbol{\nabla}^N = \boldsymbol{e}_i \partial/\partial y_i^N$) and let $\dot{\boldsymbol{u}}$ be rate of deformation (velocities) from C_N. We define the rate of displacement gradient $\dot{\boldsymbol{e}} = (\boldsymbol{\nabla}^N \dot{\boldsymbol{u}})^T$ and write $\dot{\boldsymbol{e}} = \dot{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\omega}}$ where $\dot{\boldsymbol{\varepsilon}}[\dot{\varepsilon}_{ij} = \frac{1}{2}(\partial \dot{u}_i/\partial y_j^N + \partial \dot{u}_j/\partial y_i^N)]$ is the symmetric UL strain-rate and $\dot{\boldsymbol{\omega}}[\dot{\omega}_{ij} = \frac{1}{2}(\partial \dot{u}_i/\partial y_j^N - \partial \dot{u}_j/\partial y_i^N)$ is the skew-symmetric spin-rate. Let $\dot{\boldsymbol{\tau}}$, and $\dot{\boldsymbol{\sigma}}(\equiv J^N[\dot{\boldsymbol{\tau}} + (\dot{\boldsymbol{\varepsilon}}:\boldsymbol{I})\boldsymbol{\tau}^N])$ (where $J^N = \rho_o/\rho^N$, ρ_o and ρ_N being the mass-densities in C_o and C_N, respectively), be the substantial derivatives of the Cauchy and Kirchhoff stresses respectively. As is well-known, these stress-rates are not objective. Let $\dot{\boldsymbol{t}}$, $\dot{\boldsymbol{s}}$, and $\dot{\boldsymbol{r}}$ represent the appropriate stress rates referred to C_N; ie., for instance, $\dot{\boldsymbol{s}}\Delta t = \boldsymbol{s}_N^{N+1} - \boldsymbol{\tau}^N$ where $\boldsymbol{s}_N^{N+1}$ is the second Piola-Kirchhoff stress in C_{N+1} referred to (and measured per unit area in) C_N. It is shown in [5] that:

$$\dot{\boldsymbol{s}} = (\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{e}}.\boldsymbol{\sigma}^N - \boldsymbol{\sigma}^N.\dot{\boldsymbol{e}}^T)/J^N; \quad \dot{\boldsymbol{t}} = (\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{e}}.\boldsymbol{\sigma}^N)/J^N \tag{4a,b}$$

$$\dot{\boldsymbol{r}} = \frac{1}{2}(\dot{\boldsymbol{t}} + \dot{\boldsymbol{t}}^T + \boldsymbol{\tau}^N.\dot{\boldsymbol{\omega}} + \dot{\boldsymbol{\omega}}^T.\boldsymbol{\tau}^N) = \dot{\boldsymbol{s}} + \frac{1}{2}(\boldsymbol{\tau}^N.\dot{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\varepsilon}}.\boldsymbol{\tau}^N) \tag{5a,b}$$

Unless large elastic deformations of a dilatational nature have preceded the inelastic straining, one may, without significant error, assume that $J^N \approx 1.0$.

The equations of linear momentum balance (LMB), angular momentum balance (AMB), compatibility, and traction and displacement boundary conditions (TBC and DBC) in the UL rate formulation can be written as:

$$\text{LMB:} \quad \boldsymbol{\nabla}^N.[\dot{\boldsymbol{s}} + \boldsymbol{\tau}^N.(\boldsymbol{\nabla}^N \dot{\boldsymbol{u}})] + \rho^N \dot{\boldsymbol{B}} = \boldsymbol{0} \ \text{(or)} \ \boldsymbol{\nabla}^N.\dot{\boldsymbol{t}} + \rho^N \dot{\boldsymbol{B}} = \boldsymbol{0} \tag{6a,b}$$

AMB: $\dot{s} = \dot{s}^T$; (or) $(\nabla^N \dot{u})^T . \tau^N + \dot{t} = \dot{t}^T + \tau^N . (\nabla^N \dot{u})$ (7a,b)

or, equivalently,

$$\dot{\omega} . \tau^N + \dot{\varepsilon} . \tau^N + \dot{t} = \dot{t}^T + \tau^N . \dot{\varepsilon} + \tau^N . \dot{\omega}^T \tag{7c}$$

compatibility:

$$\dot{e} \equiv \dot{\varepsilon} + \dot{\omega} = (\nabla^N \dot{u})^T; \text{ (or) } \dot{\varepsilon} = (\nabla^N \dot{u}) + (\nabla^N \dot{u})^T \tag{8a,b}$$

TBC: $n^* . [\dot{s} + \tau^N . (\nabla^N \dot{u})] \equiv n^* . [\dot{t}] = \dot{\bar{t}}$ at $S_{\sigma N}$ (9)

DBC: $\dot{u} = \dot{\bar{u}}$ at S_{uN} (10)

Let us suppose, for the moment, that the consitutive law for a rate-dependent material can be expressed (as shown later in this paper) in terms of certain rate-potentials, as:

$$\dot{s} = \partial \dot{W} / \partial \dot{\varepsilon}; \quad \dot{t} = \partial \dot{U} / \partial \dot{e}^T; \quad \dot{r} = \partial \dot{Q} / \partial \dot{\varepsilon} \tag{11a-c}$$

We consider the Legendre (contact) transformations of the type:

$$\dot{s} : \dot{\varepsilon} - \dot{W}(\dot{\varepsilon}) = \dot{S}^*(\dot{s}); \quad \dot{t}^T : \dot{e} - \dot{U}(\dot{e}) = \dot{E}^*(\dot{t});$$
$$\dot{r} : \dot{\varepsilon} - \dot{Q}(\dot{\varepsilon}) = \dot{R}^*(\dot{r}) \tag{12a-c}$$

such that

$$\partial \dot{S}^* / \partial \dot{s} = \dot{\varepsilon}; \quad \partial \dot{E}^* / \partial \dot{t} = \dot{e}^T; \quad \partial \dot{R}^* / \partial \dot{r} = \dot{\varepsilon}$$

As discussed in [1-6], the AMB conditions are embedded in the structure of $\dot{W}$ and $\dot{U}$. As shown in [5], the complementary energy principles, and the Hellinger-Reissner type principles, involving the stress-rates $\dot{s}$, $\dot{t}$, and $\dot{r}$ are as below. In each case the functional whose stationary condition is the principle in question is given. The respective functionals are, denoted by π with the subscript C and HR denoting complementary and Hellinger-Reissner type functionals, respectively.

$$\pi_C(\dot{u}, \dot{s}) = \int_{V_N} \{ -\dot{S}^*(\dot{s}) - \tfrac{1}{2} \tau^N : [(\nabla^N \dot{u}) . (\nabla^N \dot{u})^T] \} dv + \int_{S_{UN}} \dot{t} . \dot{\bar{u}} ds \tag{13}$$

$$\pi_{HR}(\dot{u}, \dot{s}) = \int_{V_N} \{ -\dot{S}^*(\dot{s}) + \rho^N \dot{B} . \dot{u} + \tfrac{1}{2} \tau^N : [(\nabla^N \dot{u}) . (\nabla^N \dot{u})^T]$$
$$+ \tfrac{1}{2} \dot{s} : [(\nabla^N \dot{u}) + (\nabla^N \dot{u})^T] \} dv - \int_{S_{\sigma N}} \dot{\bar{t}} . \dot{u} ds - \int_{S_{uN}} \dot{t} . (\dot{u} - \dot{\bar{u}}) ds \tag{14}$$

$$\pi_C(\dot{t}) = \int_{V_N} -\dot{E}^*(\dot{t}) dv + \int_{S_{uN}} \dot{t} . \dot{\bar{u}} ds \tag{15}$$

$$\pi_{HR}(\dot{t}, \dot{u}) = \int_{V_N} \{ -\dot{E}^*(\dot{t}) dv - \rho^N \dot{B} . \dot{u} + \dot{t}^T : [(\nabla^N \dot{u})^T] \} dv \quad + \text{contd.}$$

$$- \int_{S_{\sigma N}} \dot{\underset{\sim}{t}} . \dot{\underline{u}} ds - \int_{S_{uN}} \dot{\underline{t}} . (\dot{\underline{u}} - \dot{\bar{\underline{u}}}) ds \tag{16}$$

$$\pi_C(\dot{\underset{\sim}{t}}, \dot{\underset{\sim}{\omega}}) = \int_{V_N} \{-\dot{R}^*(\dot{\underset{\sim}{t}}) + \frac{1}{2}\underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\omega}}^T . \dot{\underset{\sim}{\omega}}) - \dot{\underset{\sim}{t}}^T : \dot{\underset{\sim}{\omega}}\} dv + \int_{S_{uN}} \dot{\underline{t}} . \dot{\bar{\underline{u}}} ds \tag{17}$$

$$\pi_{HR}(\dot{\underset{\sim}{t}}; \dot{\underset{\sim}{\omega}}; \dot{\underline{u}}) = \int_{V_N} \{-\dot{R}^*(\dot{\underset{\sim}{t}}) + \frac{1}{2}\underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\omega}}^T . \dot{\underset{\sim}{\omega}}) - \rho^N \dot{\underline{B}} . \dot{\underline{u}} + \dot{\underset{\sim}{t}}^T : [\underline{\nabla}^N \dot{\underline{u}}]^T - \dot{\underset{\sim}{t}}^T : \dot{\underset{\sim}{\omega}}\} dv$$

$$- \int_{S_{\sigma N}} \dot{\bar{\underline{t}}} . \dot{\underline{u}} ds - \int_{S_{uN}} \dot{\underline{t}} . (\dot{\underline{u}} - \dot{\bar{\underline{u}}}) ds \tag{18}$$

In the above V_N, $S_{\sigma N}$, and S_{uN} are the volume, prescribed-traction boundary, and prescribed displacement boundary, respectively, of the solid in C_N; and ρ^N and $\dot{\underline{B}}$, are respectively, the mass-density and rate-of-body-force in C_N. The above functionals are valid, in general, for non-conservative (deformation-dependent) surface tractions. If A is the set of conditions that are satisfied a priori, and B is the set of those conditions that are satisfied a posteriori in the variational principle, for each of the above functionals the sets A and B are as follows: (*i*) Eq. (13): Set A(Eqs. 6a, 7a, and 9), set B(Eqs. 8b, and 10) (*ii*) Eq. (14): Set A (the existence of $\dot{S}^*$ such that $\partial \dot{S}^*/\partial \dot{\underset{\sim}{s}} = \dot{\underset{\sim}{\varepsilon}}$, and Eq. 7a), set B(Eqs. 6a, 8b, 9, and 10). (*iii*) Eq. (17): Set A(Eqs. 6b, 9, the definition of $\dot{\underset{\sim}{r}}$ as in Eq. 5a and that $\dot{\underset{\sim}{\omega}}$ is skew-symmetric), set B(Eqs. 7c, 8a, and 10); (*iv*) Eq. 18: Set A(the definition of $\dot{\underset{\sim}{r}}$ as in Eq. (5a), and that $\dot{\underset{\sim}{\omega}}$ is skew symmetric), set B(6b, 7c, 8a, 9, and 10).

The complementary, and Hellinger-Reissner type principles as through Eqs. (17) and (18) were first stated by Atluri [5]. The general invalidity of the principles through Eq. (15) which was alluded to by Hill [11], and Eq. (16), were discussed in [5]. Eventhough, Eqs. (13) and (14), and the attendant variational principles, may be viewed as being consistent, the limitations of practical applicability of these are discussed in [5]. Especially, in the application of Eq. (13), the need to select a symmetric $\dot{\underset{\sim}{s}}$, such that Eq. (6a) (which involved coupling with $\dot{\underline{u}}$) is satisfied, a priori, is not an altogether easy proposition. Several interesting ways of satisfying Eq. (6a), and of application of Eq. (13), were discussed by Atluri [6,7,12].

However, the complementary principle of Eq. (17), introduced in [5], has several attractive features for practical application: (*i*) the LMB, Eq. (6b) can be easily satisfied by setting: $\dot{\underset{\sim}{t}} = \underset{\sim}{\nabla}^N x \underset{\sim}{\psi} + \dot{\underset{\sim}{t}}^p$ where ψ are first-order (once differentiable) stress functions (*ii*) $\dot{\underset{\sim}{\omega}}$ can be selected to be skew symmetric, by setting $\dot{\omega}_{ij} = e_{kij}\Omega_k$ where e_{kij} is an alternating tensor). In general, even in a TL formulation, the constraint $\underset{\sim}{\alpha} . \underset{\sim}{\alpha}^T = I$ is easily met by taking $\underset{\sim}{\alpha}$ to be a function of the 3 Euler-angles of rigid rotation [4]. *In the case of plates and shells,* the concepts of a finite-rotation vector, as discussed later, may be employed.

The application of the complementary energy principle as through Eq. (17), and its TL rate counterpart, is illustrated later in this paper.

RATE POTENTIALS FOR RATE-SENSITIVE MATERIALS:

As discussed in [5], and elsewhere, the principle of objectivity is met, in writing the rate constitutive law of the material, by postulating the constitutive relation between the objective strain-rate $\dot{\underset{\sim}{\varepsilon}}$ and the objective co-rotational (or also at times referred to as the Zaremba, or the rigid-body or the Jaumann) rate of Kirchhoff stress, denoted here by $\dot{\underset{\sim}{\sigma}}^*$. It is well-known that,

$$\dot{\underset{\sim}{\sigma}}^* = \dot{\underset{\sim}{\sigma}} - \dot{\underset{\sim}{\omega}} . \underset{\sim}{\sigma}^N - \underset{\sim}{\sigma}^N . \dot{\underset{\sim}{\omega}}^T \tag{19}$$

Thus, in view of Eqs. (4-5),

$$\dot{\underset{\sim}{s}} = (\dot{\underset{\sim}{\sigma}}^* - \dot{\underset{\sim}{\varepsilon}} . \underset{\sim}{\sigma}^N - \underset{\sim}{\sigma}^N . \dot{\underset{\sim}{\varepsilon}})/J^N; \quad \dot{\underset{\sim}{t}} = (\dot{\underset{\sim}{\sigma}}^* - \dot{\underset{\sim}{\varepsilon}} . \underset{\sim}{\sigma}^N - \underset{\sim}{\sigma}^N . \dot{\underset{\sim}{\omega}})/J^N$$

$$\dot{\underset{\sim}{r}} = [\dot{\underset{\sim}{\sigma}}^* - \frac{1}{2}(\dot{\underset{\sim}{\varepsilon}} . \underset{\sim}{\sigma}^N + \underset{\sim}{\sigma}^N . \dot{\underset{\sim}{\varepsilon}})]/J^N \tag{21}$$

Thus, if $\dot{V}$ is the postulated rate potential for $\dot{\underset{\sim}{\sigma}}^*$ such that $\partial\dot{V}/\partial\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{\sigma}}^*$, we can define:

$$J^N\dot{W} = \dot{V} - \underset{\sim}{\sigma}^N : (\dot{\underset{\sim}{\varepsilon}} . \dot{\underset{\sim}{\varepsilon}}); \quad \partial\dot{W}/\partial\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{s}} \tag{22}$$

$$J^N\dot{U} = \dot{W} + \frac{1}{2}\underset{\sim}{\sigma}^N : (\dot{\underset{\sim}{e}}^T . \dot{\underset{\sim}{e}}); \quad \partial\dot{U}/\partial\dot{\underset{\sim}{e}}^T = \dot{\underset{\sim}{t}} \tag{23}$$

$$J^N\dot{Q} = \dot{V} - \frac{1}{2}\underset{\sim}{\sigma}^N : (\dot{\underset{\sim}{\varepsilon}} . \dot{\underset{\sim}{\varepsilon}}); \quad \partial\dot{Q}/\partial\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{r}} \tag{24}$$

From these, one can establish $\dot{S}^*(\dot{\underset{\sim}{s}})$, $\dot{E}^*(\dot{\underset{\sim}{t}})$ and $\dot{R}^*(\dot{\underset{\sim}{r}})$ as defined in Eqs. (12a-c). Thus, we focus attention on the potential $\dot{V}$.

It is worth noting that for materials with *rate-independent* constitutive laws, such as classical elastic-plastic materials the derivative $(\dot{})$ is considered to be with respect to a fictition time. However, for rate-sensitive materials, such as elasto-viscoplastic and creeping materials, the derivative $(\dot{})$ is w.r.t. to natural time.

For rate-independent classical elastic-plastic materials, Hill [13] presented the postulation:

$$\dot{V} = \frac{1}{2} L_{ijkl}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{kl} - \frac{\alpha}{g}(\lambda_{kl}\dot{\varepsilon}_{kl})^2 \tag{25}$$

where L_{ijkl} is a +ve definite symmetric (under $ij \leftrightarrow kl$ interchange) tensor of instantaneous elastic modulii, $\alpha = 1$ or 0 according to whether $\lambda_{kl}\,\dot{\varepsilon}_{kl}$ is positive or negative, g is a scalar related to the measure of hardening, and λ_{ij} is a tensor normal to the interface between elastic and plastic domain in the $\dot{\varepsilon}_{kl}$ space. Prandtl-Reuss type rate equations of type (25) for classical isotropically hardening materials can easily be derived, formally, to be [5]:

$$\dot{\underset{\sim}{\sigma}}^* = 2\mu\dot{\underset{\sim}{\varepsilon}} + \lambda(\dot{\underset{\sim}{\varepsilon}}:\underset{\sim}{I})\underset{\sim}{I} - 12\alpha\mu^2 \frac{(\dot{\underset{\sim}{\varepsilon}}:\underset{\sim}{\sigma}')\underset{\sim}{\sigma}'}{(\underset{\sim}{\sigma}':\underset{\sim}{\sigma}')[6\mu+2(\partial F_o/\partial W^P)]} \tag{26}$$

where, λ and μ are Lame' constants, $\underset{\sim}{\sigma}' \equiv \underset{\sim}{\sigma} - 1/3(\underset{\sim}{\sigma}:\underset{\sim}{I})\underset{\sim}{I}$ is deviatoric Kirchhoff stress, and the yield-surface is represented by $F = [3J_2(\underset{\sim}{\sigma}')]^{\frac{1}{2}} - F_o = 0$.

Here $F_o = F_o(W^p)$ where $W^p = \int \underset{\sim}{\sigma}:\dot{\underset{\sim}{\varepsilon}}^p dt$; and $J_2(\underset{\sim}{\sigma}') = (1/2)(\underset{\sim}{\sigma}':\underset{\sim}{\sigma}')$.

A rate-sensitive constitutive law of a considerable generality, as given by Perzyna [14], can be easily written for finite strains, when an associative flow-rule is used, as:

$$\dot{\underset{\sim}{\varepsilon}}^a = \gamma\langle\phi(F)\rangle\frac{\partial F}{\partial\underset{\sim}{\sigma}} \tag{27}$$

where $\langle\phi\rangle$ denotes a specific function, such that $\langle\phi\rangle = \phi(F)$ for $F>0$, and $\phi=0$ for $F\leq 0$. The parameter γ is called the fluidity parameter and $\dot{\underset{\sim}{\varepsilon}}^a$ is the general anelastic strain. It has been shown by Zienkiewicz and Coworkers [15], and Argyris and Coworkers [16] that classical, rate-independent elasto-plastic solutions can be obtained from the above theory, when (*i*) either $\gamma\to\infty$ or (*ii*) a stationary solution of the visco-plastic flow is sought. Various forms of ϕ were reviewed by Perzyna [14]. For the Hencky-Mises-Huber yield criterion, one can define F to be:

$$F = [3J_2(\underset{\sim}{\sigma}')]^{\frac{1}{2}} - F_o \equiv \sigma_{eq} - F_o \tag{28}$$

where σ_{eq} is the equivalent Kirchhoff stress. A simple choice for $\phi(F)$ can be:

$$\phi(F) = F^n \tag{29}$$

With Eq. (29), the viscoplastic strain rate as in Eq. (27) can be seen to correspond to the well-known Norton's power law for steady-state creep when $F_o \to 0$. We now derive rate potentials $\Delta V(=\dot{V}\Delta t), \Delta W$, etc. for the viscoplastic constitutive laws given by Eqs. (27-29).

Let at times t_N and $t_N+\Delta t$, the Kirchhoff stresses be $\underset{\sim}{\sigma}^N+\overset{\circ}{\underset{\sim}{\sigma}}\Delta t \equiv \underset{\sim}{\sigma}^N+\Delta\underset{\sim}{\sigma}$, respectively, where $\overset{\circ}{\underset{\sim}{\sigma}}$ is the substantial derivative. The inelastic strain-rates corresponding to Eqs. (27-29) at times t_N and $t_N+\Delta t$, are given, respectively, by:

$$\dot{\underset{\sim}{\varepsilon}}^a_N \equiv (\dot{\underset{\sim}{\varepsilon}}^a)_N = \gamma F^N(\partial F/\partial\underset{\sim}{\sigma}) = \gamma(3/2\sigma_{eq})(\sigma_{eq}-F_o)^n\underset{\sim}{\sigma}' \tag{30}$$

In the above $\underset{\sim}{\sigma}' \equiv (\underset{\sim}{\sigma}^N)'$ (ie., superscript N dropped for convenience) and $\sigma^2_{eq} = (3/2)\underset{\sim}{\sigma}':\underset{\sim}{\sigma}'$. Likewise at a time Δt later,

$$\dot{\underset{\sim}{\varepsilon}}^a_{N+1} = \gamma(3/2)(\sigma_{eq}+\Delta\sigma_{eq})^{-1}(\sigma_{eq}+\Delta\sigma_{eq}-F_o)^n(\underset{\sim}{\sigma}'+\Delta\underset{\sim}{\sigma}') \tag{31}$$

By straight forward algebra, it can be shown that,

$$\dot{\underset{\sim}{\varepsilon}}^a_{N+1} = \dot{\underset{\sim}{\varepsilon}}^a_N+\gamma(3/2\sigma_{eq})\{-(\Delta\sigma_{eq}/\sigma_{eq})(\sigma_{eq}-F_o)^n\underset{\sim}{\sigma}'+(\sigma_{eq}-F_o)^n\Delta\underset{\sim}{\sigma}'$$

$$+ n(\sigma_{eq}-F_o)^{n-1}\underset{\sim}{\sigma}'\Delta\sigma_{eq}\} \tag{32}$$

However, it can be easily shown that

$$\Delta\sigma_{eq} = (3/2)[(\underset{\sim}{\sigma}':\Delta\underset{\sim}{\sigma}')/\sigma_{eq}] \tag{33}$$

Using (33) in (32) we obtain:

$$\dot{\underset{\sim}{\varepsilon}}^a_{N+1} = \dot{\underset{\sim}{\varepsilon}}^a_N + \underset{\approx}{V}:\Delta\underset{\sim}{\sigma} \tag{34}$$

wherein the definition of $\underset{\approx}{V}$ is apparent. Since $\Delta\underset{\sim}{\sigma}$ can include the effects of pure spin between t_N and t_{N+1}, one can replace Eq. (34) by:

$$\dot{\underset{\sim}{\varepsilon}}^a_{N+1} = \dot{\underset{\sim}{\varepsilon}}^a_N + \underset{\approx}{V}:\Delta\underset{\sim}{\sigma}^* \tag{35}$$

where $\Delta\underset{\sim}{\sigma}^* = \overset{\circ}{\underset{\sim}{\sigma}}{}^*\Delta t$ is the corotational increment of Kirchhoff stress. Now, the corotational rate $\overset{\circ}{\underset{\sim}{\sigma}}{}^*$ can be written as:

$$\overset{\circ}{\underset{\sim}{\sigma}}{}^* = \underset{\approx}{L}_e:(\dot{\underset{\sim}{\varepsilon}}-\dot{\underset{\sim}{\varepsilon}}^a) \tag{36}$$

Where $\underset{\approx}{L}_e$ is the tensor of instantaneous elastic moduli.

$$\text{Thus,}\quad \Delta\underset{\sim}{\sigma}^* = \underset{\approx}{L}_e:\Delta\underset{\sim}{\varepsilon} - L_e:\int_{t_N}^{t_N+\Delta t}\dot{\underset{\sim}{\varepsilon}}^a dt \tag{37}$$

one may use the approximation,

$$\dot{\underset{\sim}{\varepsilon}}^{a} = (1-\beta)\dot{\underset{\sim}{\varepsilon}}^{a}_{N} + \beta\dot{\underset{\sim}{\varepsilon}}^{a}_{N+1} \qquad t_N \le t \le t_N+\Delta t \tag{38}$$

when Eqs. (35) and (38) are used, Eq. (37) becomes,

$$\Delta\underset{\sim}{\sigma}^{*} = \underset{\approx}{L}_{e}:\Delta\underset{\sim}{\varepsilon} - \Delta t\underset{\approx}{L}_{e}:(\dot{\underset{\sim}{\varepsilon}}^{a}_{N}+\beta\underset{\approx}{V}\Delta\underset{\sim}{\sigma}^{*}) \tag{39}$$

From which, upon rearranging terms,

$$\Delta\underset{\sim}{\sigma}^{*} = \underset{\approx}{M}:(\Delta\underset{\sim}{\varepsilon}-\Delta\bar{\underset{\sim}{\varepsilon}}_{a}) \tag{40}$$

wherein the definition of $\underset{\approx}{M}$ is apparent, and $\Delta\bar{\underset{\sim}{\varepsilon}}_{a}$ is known and is given by: $\Delta\bar{\varepsilon}_{a} = \Delta t\dot{\varepsilon}^{a}_{N}$. From Eq. (40) one can immediately write

$$\Delta V = \frac{1}{2}M_{ijk\ell}\Delta\varepsilon_{ij}\Delta\varepsilon_{k\ell} - M_{ijk\ell}\Delta\bar{\varepsilon}_{aij}\Delta\varepsilon_{k\ell} \tag{41}$$

From Eq. (41), the potentials ΔW, ΔU, and ΔQ can easily be obtained through Eqs. (22-24).

It is noted that Wang [17] attempted to derive a relation between $\Delta\underset{\sim}{s}$, $\Delta\underset{\sim}{\varepsilon}$, $\Delta\bar{\underset{\sim}{\varepsilon}}_{a}$, and hence ΔW directly. However, this derivation appears to be in error, since, among other reasons, the transformation between the deviatoric part of $\underset{\sim}{s}$ and the deviatoric part of $\underset{\sim}{\tau}$ was assumed to be the same as that between $\underset{\sim}{s}$ and $\underset{\sim}{\tau}$ themselves.

All the above developments for the UL rate formulation can be converted to a TL rate formulation by noting the relations [5]:

$$\underset{\sim}{E}' = (\underset{\sim}{F}^{N})^{T}.\dot{\underset{\sim}{\varepsilon}}.\underset{\sim}{F}^{N};\ \underset{\sim}{e}' = \dot{\underset{\sim}{e}}.\underset{\sim}{F}^{N};\ \ \underset{\sim}{s}' = J^{N}(\underset{\sim}{F}^{N})^{-1}.\dot{\underset{\sim}{s}}.(\underset{\sim}{F}^{N})^{-T}$$

$$\underset{\sim}{t}' = J^{N}(\underset{\sim}{F}^{N})^{-1}.\dot{\underset{\sim}{t}};\ \ \underset{\sim}{r}' = (1/2)[\underset{\sim}{t}^{N}.\underset{\sim}{\alpha}'+\underset{\sim}{\alpha}'^{T}.\underset{\sim}{t}^{NT}+\underset{\sim}{t}'.\underset{\sim}{\alpha}^{N}+\underset{\sim}{\alpha}^{NT}.\underset{\sim}{t}'^{T}] \tag{42}$$

where $\underset{\sim}{E}'$ is TL rate of Gree-strain, and $\underset{\sim}{e}'$, $\underset{\sim}{s}'$, $\underset{\sim}{t}'$, and $\underset{\sim}{r}'$ are TL rates of $\underset{\sim}{e}$, $\underset{\sim}{s}$, $\underset{\sim}{t}$, and $\underset{\sim}{r}$ respectively. Now $\underset{\sim}{\alpha}'$ is subject to the constraint that $\underset{\sim}{\alpha}^{NT}.\underset{\sim}{\alpha}'$ is skew-symmetric.

UNIQUENESS & STABILITY CRITERION:

In the present paper, the application of the rate complementary energy principle as embodied in Eq. (17), or its TL counterpart, will be used in some computational studies. In the direct application of Eq. (17), the assumed stress field $\dot{\underset{\sim}{t}}$ must not only satisfy the LMB condition within each element, but also the traction reciprocity condition at the interelement boundary, viz., $(\underset{\sim}{n}^{*}.\dot{\underset{\sim}{t}})^{+}+(\underset{\sim}{n}^{*}.\dot{\underset{\sim}{t}})^{-} = 0$ at ρ_{mN} (where + and -, respectively, indicate the two sides of ρ_{mN}, the interface between mth and (m+1)th elements in C_N). In the present work this interelement condition is introduced as a postericri constraint, through a Lagrange multiplier $\dot{\underset{\sim}{u}}_{\rho}$ at ρ_{mN}, in the functional of Eq. (17), thus leading to a 'hybrid' finite element method. The thus modified functional is:

$$\pi_{HS}(\dot{\underset{\sim}{\omega}},\dot{\underset{\sim}{t}},\dot{\underline{\underline{u}}}_\rho)=\sum_m\{\int_{V_{mN}}[-\dot{R}^*(\dot{\underset{\sim}{r}})+(1/2)\underset{\sim}{\tau}^N:(\dot{\underset{\sim}{\omega}}^T.\dot{\underset{\sim}{\omega}})-\dot{\underset{\sim}{t}}^T.\dot{\underset{\sim}{\omega}}]dv$$

$$+\int_{S_{umN}}(\underline{n}^*.\dot{\underset{\sim}{t}}).\dot{\underline{\underline{u}}}ds+\int_{\rho_{mN}}(\underline{n}^*.\dot{\underset{\sim}{t}}).\dot{\underline{\underline{u}}}_\rho ds \qquad (43)$$

At the point of bifurcation, or instability, from the concept of adjacent compatible states, the following criterion can be shown to hold:

$$\pi_{HS}=0 \text{ and } \delta\pi_{HS}=0 \qquad (44)$$

with the constraints:

$$\underline{\nabla}^N.\dot{\underset{\sim}{t}}=0;\ \dot{\underset{\sim}{\omega}}=-\dot{\underset{\sim}{\omega}}^T;\ \underline{n}^*.\dot{\underset{\sim}{t}}=0 \text{ at } S_{\sigma mN};$$

and $\dot{\underline{\underline{u}}}=0$ at S_{umN} (45)

In the case of linear pre-buckling states, the above criterion reduces to an eigen-value problem, with the eigen-value depending on $\underset{\sim}{\tau}^N$.

EXAMPLE PROBLEMS:

Necking of an Initially Perfect, Plane-Strain, Elastic-Plastic Bar.

Cowper and Onat [18] examined the above bifurcation necking problem, for a bar of rigid-plastic work-hardening material, under uniform tension applied at the ends of the bar. Mises' yield, and isotropic hardening criteria were used [18]. In [18] only the eigen-value problem for the applied tension at which necking would initiate in the bar was treated, but the phenomenon of post-bifurcation necking was not treated in [18]. $L_o(L)$ and $B_o(B)$ are the initial (current) length and width of the bar respectively. If y_1^N and y_2^N are the current cartesian coordinates of a material particle, the boundary conditions are: (*i*) at $y_1^N=0$: $\dot{u}_1=-V$; $\dot{\underline{t}}_2=0$; (*ii*) at $y_1^N=L$: $\dot{u}_1=+V$; $\dot{\underline{t}}_2=0$; (*iii*) at $y_2^N=\pm(B/2)$: $\dot{\underline{t}}_1=\dot{\underline{t}}_2=0$.

In the present analysis, the complementary energy formulation based on the TL rate equivalent of Eq. (17) was used. The problem parameters used are: $(B_o/L_o)=(1/3)$; τ_y (yield stress) = $4\text{x}10^4$ psi; the true stress versus logarithmic strain ($\ln(l/l_o)$) curve was assumed to be bilinear, with the two slopes, $E=10^7$ psi, and $h = 5\text{x}10^4$ psi. The notation

$\eta = (L-L_o)L_o$; is used. The eigen-value solution for η at bifurcation-necking for the perfect bar, as obtained in [18] for the present linear-hardening but rigid-plastic material, is $\eta^c = 0.48$.

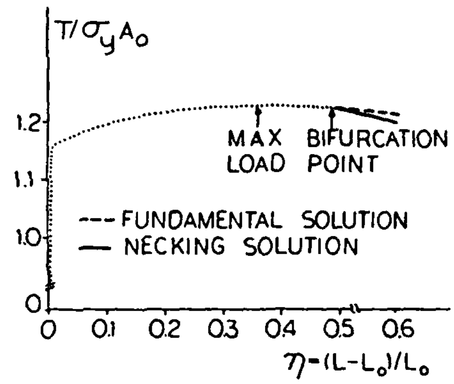

Figure 1

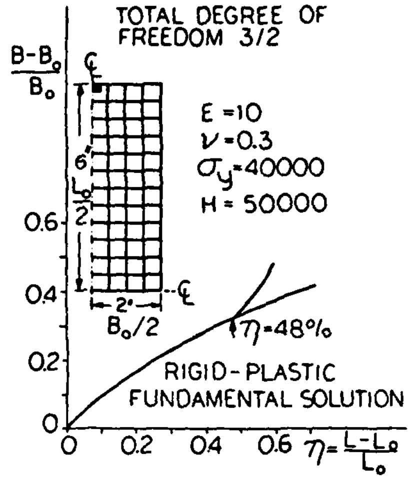

Figure 2

The variation of total applied load with η is shown in Fig. 1, from which it is seen that necking starts at $\eta^c = 0.482$ (which is in excellent agreement with the value of $\eta^c = 0.48$ of [18]). It is also seen that necking starts after the maximum load is attained. The convergence of η^c with the finite element mesh has been reported elsewhere [8,9], with the mesh as shown in Fig. 2, which is used to obtain the remainder of the reported results, being the finest mesh reported in [8]. The variation of the width reduction ratio, $(B-B_o)/B_o$ is shown in Fig. 2, from which it is seen that at $\eta = \eta^c$ the width reduction becomes much more pronounced as compared to the rigid-plastic fundamental solution.

The variation of δ/L_o (with δ being defined as the difference of widths at loading edge and the necking sections, respectively) with η is shown in Fig. 3. The slope of this curve at the beginning of necking, viz, $\eta = \eta^c = 0.48$, was obtained in an asymptotic analysis in [18]. The present result for this initial slope is in excellent agreement with [18]. However as necking develops, the slope $(\partial\eta/\partial\delta)$ decreases from the initial value at $\eta = \eta^c$, which appears to be in contradiction with the result of McMeeking and Rice [19]. The necked profile of the bar for $\eta \geq \eta^c$, are shown in Fig. 4.

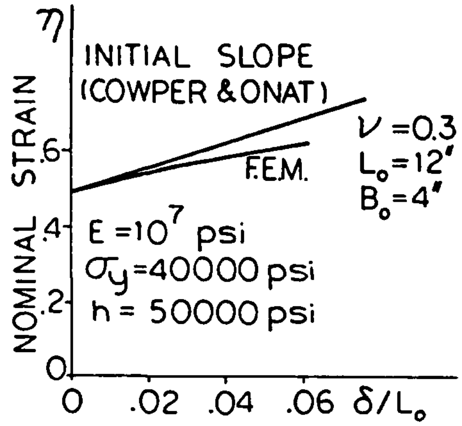

Figure 3

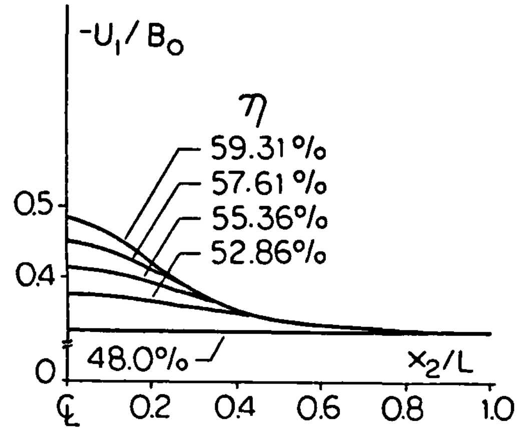

Figure 4

The necked profiles of the bar, and the progressive development of unloaded regions (shaded) are shown in Fig. 5 at various values of η. Note that unloading begins at the center of the loaded face of the bar at $\eta = \eta^c = 0.482$. Finally the distribution of Cauchy stresses, τ_{11} (in the direction of loading) τ_{22} (in the width loading), and τ_{33} (in the thickness direction of this plane-strain specimen), at the neck ($y_1^N = L/2$) are shown in Fig. 6. These results are in excellent qualitative agreement with those of Needleman [20] who also analyses the necking and post-necking problem of an initially perfect cylindrical bar. It is noted that

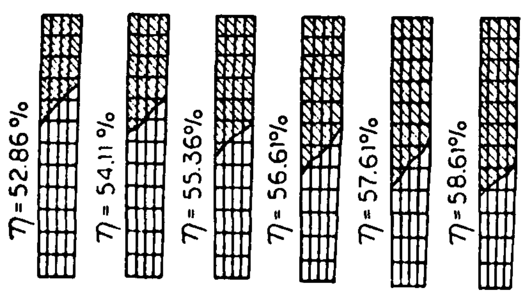

Figure 5

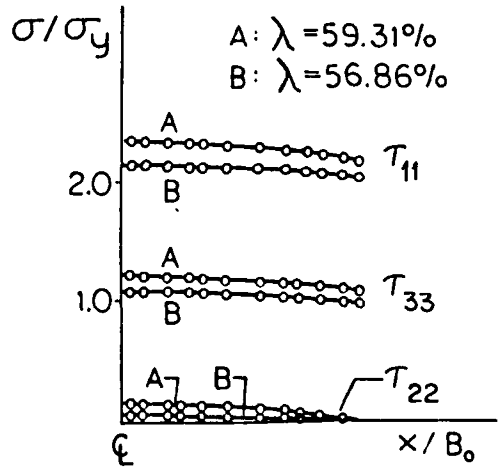

Figure 6

the problem of necking of an elastic-plastic bar, with initial imperfections, was also analysed by Osias [21], McMeeking and Rice [19], and Nemat-Nasser and Taya [21]. It is observed that the finite element meshes used in [19-21] are very much finer than the one used presently. Eventhough a precise mathematical statement as to this appears impossible, the above comparison of the mesh-sizes appears to indicate the relative advantages of the present complementary energy method.

Finite Deformation and Post-Buckling Analyses of a Thin Plate:

We consider large deformations (large rotations and large stretches) of a thin plate made of a semi-linear isotropic material (ie., a material exhibiting a linear relation between the stretch tensor $\underset{\sim}{h}$ and its conjugate stress-measure, the symmetrized Biot-Lure' or Jaumann tensor $\underset{\sim}{r}$). We invoke the well-known Kirchhoff-Love type plausible deformation hypotheses that a normal to the midplane of the initially flat undeformed plate remains normal to the deformed midplane and that there is no thickness stretch. In order to derive a consistent complementary energy principle for this constrained deformation problem, we start with the general (Hu-Washizu type) variational principle involving $\underset{\sim}{h}$, $\underset{\sim}{\alpha}$, $\underline{u}$, and $\underset{\sim}{t}$ as variables.

By introducing the appropriate approximations to all these variables, we derive a general variational principle for the plate problem; from this we proceed to construct a consistent complementary energy principle for the plate problem. It is shown in [1] that a general functional, for the three-dimensional finite elasticity, whose stationary conditions lead to all the appropriate field equations, is given by:

$$\pi_G(\underline{u},\underset{\sim}{h},\underset{\sim}{\alpha},\underset{\sim}{t}) = \int_{V_o} \{W(\underset{\sim}{h}) - \rho_o \underline{g}.\underline{u} + \underset{\sim}{t}^T:[(\underset{\sim}{I}+\underline{\nabla}\underline{u})^T$$

$$- \underset{\sim}{\alpha}.(\underset{\sim}{I}+\underset{\sim}{h})]\}dv - \int_{S_{uo}} t.(\underline{u}-\bar{\underline{u}})ds - \int_{S_{\sigma o}} \bar{\underline{t}}.\underline{u}ds \qquad (46)$$

where $W(\underset{\sim}{h})$ is the strain energy density per unit of initial volume V_o, as a function of pure stretch $\underset{\sim}{h}$; ρ_o is mass density/unit initial volume, and $\underline{g}$ are body forces/unit mass.

Let x_1, x_2 be cartesian coordinates in the mid plane, and x_3 normal to the midplane, of the plate. The position vector of an arbitrary material point in the undeformed plate is $\underline{x}=x_i\underline{e}_i$ (i=i,...3). Under the present deformation hypotheses, the position vector of the same particle after deformation is, $\underline{y} = (x_1+u_1^*)\underline{e}_1 + (x_2+u_2^*)\underline{e}_2 + u_3^*\underline{e}_3 + x_3\underline{N}$ where $\underline{N}$ is a unit normal to the deformed midplane. Further, the displacement u_i^* (i=1,2,3) are functions of x_1 and x_2 only. Thus the displacement of an arbitrary particle in the plate, is $\underline{y}-\underline{x} = u_1^*\underline{e}_1 + u_2^*\underline{e}_2 + u_3^*\underline{e}_3 + x_3(\underline{N}-\underline{e}_3)$. The base vector at an arbitrary point in the deformed plate are given by:

$$\partial \underline{y}/\partial x_\alpha = (\delta_{i\alpha}+u^*_{i,\alpha})\underline{e}_i+\underline{N},_\alpha x_3 \equiv G_\alpha \quad (\alpha=1,2;\ i=1,2,3) \qquad (47)$$

$$\partial \underline{y}/\partial x_3 = \underline{N}$$

The deformation gradient is

$$\underset{\sim}{F} = (\underline{\nabla}\underline{y})^T = \underline{G}_\alpha\underline{e}_\alpha + \underline{N}\underline{e}_3 \quad (\alpha=1,2) \qquad (48a)$$

Further, for the present kinematic hypotheses, we assume the stretch tensor to be:

$$\underset{\sim}{h} = h_{\alpha\beta}\underline{e}_\alpha\underline{e}_\beta;\ h_{\alpha\beta} = h_{\alpha\beta}(x_i) \quad [i=1,\ldots 3;\ \alpha,\ \beta=1,2] \qquad (48b)$$

From Eqs. (48a,b) it is seen:

$$\underline{N} = \underset{\sim}{F}\cdot\underline{e}_3 = [\underset{\sim}{\alpha}\cdot(\underset{\sim}{I}+\underset{\sim}{h})]\cdot\underline{e}_3 = \underset{\sim}{\alpha}\cdot\underline{e}_3 \qquad (49)$$

Thus the displacement vector can be written as:

$$\underline{u}(x_i) = u_i^*(x_\alpha)\underline{e}_i+(\underset{\sim}{\alpha}-\underset{\sim}{I})\cdot\underline{e}_3 x_3 \quad [i=1,2,3;\ \alpha=1,2] \qquad (50)$$

Further, we assume that $h_{\alpha\beta}(x_i)$ can be approximated as:

$$h_{\alpha\beta}(x_i) = h^*_{\alpha\beta}(x_\alpha) + x_3\chi_{\alpha\beta}(x_\delta) \quad [\alpha,\beta,\gamma=1,2] \qquad (51)$$

ie. $\underset{\sim}{h} = \underset{\sim}{h}^* + x_3\underset{\sim}{\chi}$

For the semilinear isotropic material we assume the constitutive law:

$$\underset{\sim}{r} = 2\mu\underset{\sim}{h} + \lambda(\underset{\sim}{h}:\underset{\sim}{I})\underset{\sim}{I} \qquad (52)$$

Since, for isotropy, $\underset{\sim}{h}$, $\underset{\sim}{g}$, and $\underset{\sim}{r}$ are coaxial, Eq. (3) becomes,

$$\underset{\sim}{r} = \underset{\sim}{t}.\underset{\sim}{\alpha} \tag{53}$$

The tensors $\underset{\sim}{t}$ and $\underset{\sim}{\alpha}$ are assumed to be:

$$\underset{\sim}{t} = t_{ij}\underline{e}_i\underline{e}_j; \quad \underset{\sim}{\alpha}=\alpha_{ij}\underline{e}_i\underline{e}_j \quad [i,j=1,2,3;] \tag{54}$$

where, $t_{ij} = t_{ij}(x_1,x_2,x_3)$; $\alpha_{ij} = \alpha_{ij}(x_1,x_2)$ (55)

Finally, the external forces distributed on the plate are assumed to be specified per unit area on the mid plane of the plate to be $g_i = g_i(x_\alpha)(i=1,..3,\ \alpha=1,2)$. When the assumptions in Eqs. (50-55) are substituted in Eq. (4-6), we find through atraight-forward algebra, that

$$\begin{aligned}
&\pi_G[u_i^*,\ h_{\alpha\beta}^*,\ \chi_{\alpha\beta},\ \alpha_{ij},\ T_{\alpha i},\ M_{\alpha i}] \\
&= \int_{S_o} \{W^*(\underset{\sim}{h}^*,\underset{\sim}{\chi}) - \underline{g}.\underline{u}^* + \overset{\wedge}{\underset{\sim}{T}}{}^T:[\underline{e}_\alpha\underline{e}_\alpha + \underline{u}^*_{,\alpha}\underline{e}_\alpha \\
&+ (\underset{\sim}{\alpha}.\underline{e}_3)\underline{e}_3 - \underset{\sim}{\alpha}.(\underset{\sim}{I}+\underset{\sim}{h})] + \overset{\wedge}{\underset{\sim}{M}}{}^T:[(\underset{\sim}{\alpha}.\underline{e}_3)_{,\alpha}\underline{e}_\alpha - \underset{\sim}{\alpha}.\underset{\sim}{\chi}]\}ds \\
&- \int_{C_u} [\underline{T}.(\underline{u}^*-\bar{\underline{u}}^*) + \underline{M}.<(\underset{\sim}{\alpha}-\bar{\underset{\sim}{\alpha}}).\underline{e}_3>]dc - \int_{C_\sigma} [\bar{\underline{T}}.\underline{u}^*+\bar{\underline{M}}.(\underset{\sim}{\alpha}-\underset{\sim}{I}).\underline{e}_3]dc
\end{aligned} \tag{56}$$

where $\overset{\wedge}{\underset{\sim}{T}} = T_{\alpha i}\underline{e}_\alpha\underline{e}_i$; $\overset{\wedge}{\underset{\sim}{M}} = M_{\alpha i}\underline{e}_\alpha\underline{e}_i$ $(\alpha=1,2;\ i=1,\dots 3)$, and,

$$T_{\alpha i} = \int_{x_3} t_{\alpha i}dx_3; \quad M_{\alpha i} = \int_{x_3} t_{\alpha i}x_3dx_3$$

and $$W^* = \int_{x_3} W dx_3 \tag{57}$$

In Eq. (56), s_o is the area of the undeformed midplane, and C_u and C_σ are the displacement and traction prescribed boundaries of s. It is seen that only $t_{\alpha i}$ enter into the above energy expression due to the presently invoked deformation assumptions. The constitutive equations and LMB conditions obtainable from Eq. (57) are:

$$\partial W^*/\partial \underset{\sim}{h}^* = \overset{\wedge}{\underset{\sim}{T}}.\underset{\sim}{\alpha} \quad \underset{\sim}{R} \text{ and } \partial W^*/\partial \underset{\sim}{\chi} = \overset{\wedge}{\underset{\sim}{M}}.\underset{\sim}{\alpha} \equiv \underset{\sim}{N} \tag{58}$$

and $$T_{\alpha i,\alpha} + g_i = 0 \quad (\alpha=1,2;\ i=1,\dots 3) \tag{59}$$

When Eqs. (58, 59) and the appropriate traction boundary conditions on $T_{\alpha i}$ are satisfied a priori, one can eliminate from Eq. (56), (*i*) $\underset{\sim}{h}^*$ and $\underset{\sim}{\chi}$ through the usual contact transformations and by establishing a complementary energy density R^* such that $\partial R^*/\partial \underset{\sim}{R} = \underset{\sim}{h}^*$ and $\partial R^*/\partial \underset{\sim}{N} = \underset{\sim}{\chi}$, (*ii*) $\underline{u}^*$ through satisfying (53) a priori. When this is done, we obtain a complementary energy functional:

$$\pi_C(\underset{\sim}{\alpha},\ \underset{\sim}{R},\ \underset{\sim}{N}) = \int_s -R^*(\underset{\sim}{R},\underset{\sim}{N}) + \hat{\underset{\sim}{T}}^T:[\underline{e}_\alpha\underline{e}_\alpha+(\underset{\sim}{\alpha}\cdot\underline{e}_3)\underline{e}_3-\underset{\sim}{\alpha}]$$

$$+ \underset{\sim}{M}^T:[(\underset{\sim}{\alpha}\cdot\underline{e}_3)_{,\alpha}\underline{e}_\alpha]\,ds - \int_{C_u}(\underline{T}\cdot\bar{\underline{u}}^*+\underline{M}\cdot\langle(\underset{\sim}{\alpha}-\underset{\sim}{\alpha}^-)\cdot\underline{e}_3\rangle)dc$$

$$- \int_{C_\sigma}\bar{\underline{M}}\cdot(\underset{\sim}{\alpha}-\underset{\sim}{I})\cdot\underline{e}_3 dc. \qquad (60)$$

In Eq. (60), $\underset{\sim}{\alpha}$ is required to be orthogonal and further $\underset{\sim}{\alpha}$ is, as assumed in Eq. (55), a function only of x_1 and x_2. Also the variables $\underset{\sim}{R}$ and $\underset{\sim}{N}$ in Eq. (60) are assumed to be defined in terms of $\hat{\underset{\sim}{T}}$, $\hat{\underset{\sim}{M}}$, and $\underset{\sim}{\alpha}$ as in Eq. (58). To assume an orthogonal $\underset{\sim}{\alpha}(x_i,x_2)$, the concept of a finite rotation vector [23] is useful. Let ω be the finite angle of rotation around an arbitrarily oriented unit vector $\underline{e}$ in the midplane of the plate. The finite rotation vector is defined to be:

$$\underline{\Omega} = (\sin\omega)\underline{e} \qquad (61)$$

The action of finite rotation $\underline{\Omega}$ on a vector $\underline{V}$ can be written [23] as the transformation of $\underline{V}$ to $\underline{V}^*$ as,

$$\underline{V}^* = \underline{V} + \underline{\Omega} \times \underline{V} + [\underline{\Omega}\times(\underline{\Omega}\times\underline{V}]/2\cos^2(\omega/2) \equiv \underset{\sim}{\alpha}\cdot\underline{V} \qquad (62)$$

where $$\underset{\sim}{\alpha} = \underset{\sim}{I} + \underline{\Omega} \times \underset{\sim}{I} + [(\underline{\Omega}\times\underset{\sim}{I})\cdot(\underline{\Omega}\times\underset{\sim}{I})]/2\cos^2(\omega/2) \qquad (63)$$

It can be shown that $\underset{\sim}{\alpha}$ of Eq. (63) is orthogonal, ie., $\underset{\sim}{\alpha}\cdot\underset{\sim}{\alpha}^T = \underset{\sim}{I}$.

The vector $\underline{e}$ in Eq. (61) can be written as:

$$\underline{e} = \underline{e}_1\cos\theta + \underline{e}_2\sin\theta \qquad (64)$$

Thus, the rotation tensor $\underset{\sim}{\alpha}$ of Eq. (63) is a function of two parameters: $\theta(x_1,x_2)$ and $\omega(x_1,x_2)$. The explicit expression for $\underset{\sim}{\alpha}$ can be shown to be:

$$\underset{\sim}{\alpha}=\langle 1-(1-\cos\omega)\sin^2\theta\rangle\underline{e}_1\underline{e}_1+\langle(1-\cos\omega)\sin\theta\cos\theta\rangle\underline{e}_1\underline{e}_2$$

$$+ (\sin\omega\sin\theta)\underline{e}_1\underline{e}_3 + (1-\cos\omega)\sin\theta\cos\theta\underline{e}_2\underline{e}_1$$

$$+ \langle 1-(1-\cos\omega)\cos^2\theta\rangle\underline{e}_2\underline{e}_2 - \sin\omega\cos\theta\underline{e}_2\underline{e}_3 - \sin\omega\sin\theta\underline{e}_3\underline{e}_1$$

$$+ \sin\omega\cos\theta\underline{e}_3\underline{e}_2 + \cos\omega\underline{e}_3\underline{e}_3.$$

In a Von-Karman type plate theory ω is assumed to be moderately large, such that $\cos\omega \approx 1-(\omega^2/2)$ and $\sin\omega = \omega$; while the angle θ can be assumed to be arbitrary.

Further details of the analysis of large rotations and stretches of thin plates using the complementary energy method sketched above, which are omitted here for space reasons, will

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) General variational theorems for the rate problems of rate-dependent finite strain inelasticity, in terms of the appropriate rates of the first and second Piola-Kirchhoff stress tensors, the symmetrized Biot-Lure' stress tensor, and their conjugate measures of strain-rate, are discussed. Certain new rate-complementary=energy principles, involving the rate of spin and the rate of the symmetrized Biot-Lure' stress tensor as variables, are stated for finite strain analysis of rate-sensitive materials, such as those exhibiting elasto-visco-plastic and creep behavior. Uniequeness and stability criteria for those inelastic solids, using the finite element counter-
(see back)

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) parts of the new complementary energy rate principles, are discussed. Computational studies, using the complementary energy methods, discussed herein include: (i) bifurcation necking and post-buckling analyses of initially perfect elasto-plastic bars, and (ii) post-buckling and large-deformation analyses of thin elastic plates under inplane compression and transverse bending loads.